Affine Processes and Pseudo-Differential Operators with Unbounded Coefficients

DISSERTATION

zur Erlangung des akademischen Grades

Doctor rerum naturalium
(Dr. rer. nat.)

vorgelegt

der Fakultät Mathematik und Naturwissenschaften
der Technischen Universität Dresden

von

Schwarzenberger, Michael Alois
geboren am 22.03.1984 in Schwandorf

Gutachter: Prof. Dr. René L. Schilling
Prof. Dr. Niels Jacob

Eingereicht am: 14. 03. 2016
Tag der Disputation: 12. 05. 2016

Berichte aus der Mathematik

Michael A. Schwarzenberger

Affine Processes and Pseudo-Differential Operators with Unbounded Coefficients

Shaker Verlag
Aachen 2016

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.d-nb.de.

Zugl.: Dresden, Techn. Univ., Diss., 2016

Printed in Germany.

ISBN 978-3-8440-4708-0
ISSN 0945-0882

Shaker Verlag GmbH • P.O. BOX 101818 • D-52018 Aachen
Phone: 0049/2407/9596-0 • Telefax: 0049/2407/9596-9
Internet: www.shaker.de • e-mail: info@shaker.de

Contents

Introduction

In the 1960s, Courrège and von Waldenfels proved that the generator of a Feller semigroup is represented by an integro-differential operator under some reasonable assumptions. By Fourier inversion, we immediately derive that the generator of a Feller process is a pseudo-differential operator

$$Au(x) = -q(x, D)u(x) = -(2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top \xi} q(x, \xi)\hat{u}(\xi)\,\mathrm{d}\xi, \qquad \forall u \in C_c^\infty(\mathbb{R}^d),$$

with symbol $q(x, \xi)$. Since every Markov process is associated with a semigroup, this result establishes a connection between Feller processes and symbols of pseudo-differential operators.

For a Lévy process, this relation is quite natural. It is well-known that a Lévy process $(L_t)_{t\geq 0}$ is characterized by the characteristic exponent ψ_L,

$$\mathbb{E}^x(\mathrm{e}^{\mathbf{i}(L_t - x)^\top \xi}) = \mathrm{e}^{-t\psi_L(\xi)},$$

and that the generator of a Lévy process is given by

$$Au(x) = -(2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathbf{i}x^\top \xi} \psi_L(\xi)\hat{u}(\xi)\,\mathrm{d}\xi.$$

Hence, the characteristic exponent and the symbol of the pseudo-differential operator coincide.

Many results for Lévy processes are based on the characteristic exponent. In other words, it is possible to deduce properties of a Lévy process from its symbol. In recent years, this approach has generated new insights to Feller processes, e.g. path properties, through their symbol. However, most applications require the assumption of bounded coefficients,

$$\sup_{x\in\mathbb{R}^d} |q(x, \xi)| \leq c(1 + |\xi|^2) \quad \text{for all } \xi \in \mathbb{R}^d.$$

Hence, this thesis is devoted to the study of Feller processes with unbounded coefficients through their symbol. We address this task by focusing on affine processes. This wide class of processes has a symbol with coefficients which are affine dependent on x and hence unbounded. We develop new techniques and tools to handle the affine case and then expand our results to Feller processes with unbounded coefficients.

This analytic approach to stochastic processes requires elementary harmonic analysis. Therefore, we introduce the notion and relevant results of positive and negative definite functions in Chapter 1. Making the thesis self-contained, we provide a brief exposition of Markov process, semigroups and pseudo-differential operators.

The second chapter contains several results for Feller processes whose symbols have unbounded coefficients. We start with probability estimates, the maximal inequality and an extension of the upper bound of the tail probability. With the aid of these important tools, we are able to prove a law of iterated logarithm as well as path properties by the Blumenthal-Getoor-Pruitt indices. The analysis of pseudo-differential operators with unbounded coefficients requires weighted norms, which compensate the growth of the coefficients. Section 2.3 provides essential properties of weighted norms, allowing us to characterize the domain of a pseudo-differential operator.

Since affine processes are a major example in this thesis, we thoroughly examine them on the so-called canonical state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$ in Chapter 3. An affine process is defined as a Markov process whose characteristic function is exponential-affine dependent on x. Basic examples show that in general, we have no explicit representation of this definition. However, an affine process is uniquely characterized by its admissible parameters. After establishing some basic properties and identifying the affine semigroup as a Feller semigroup, we use harmonic analysis to verify the admissibility conditions. The parameters correspond to negative definite functions which form the symbol of the pseudo-differential operator. Based on the results of Chapter 2, we give accessible proofs of the domain and cores of an affine process. In Section 3.6, we deduce several path properties through the symbol.

Chapter 4 deals with affine processes on the space of symmetric positive semidefinite matrices. The program is similar to the canonical case. However, the matrix-valued state space causes slight differences. Nevertheless, we see that the study of affine processes through their symbol is independent of the state space to a certain extent.

In Chapter 5, we look at Ornstein-Uhlenbeck processes from a potential theoretical point of view. We consider the corresponding semigroups on the L^2 space. We will see that the Ornstein-Uhlenbeck process generates an L^2 sub-Markov semigroup, and hence a Dirichlet operator. Furthermore, we are interested in the invariance and symmetry of the operator. By a criterion, based on the symbol, we calculate the invariant measure for an Ornstein-Uhlenbeck process. The symmetry of the generator of an Ornstein-Uhlenbeck process implies a functional equation. The solution of the latter shows that the symmetry requires that the process has no jumps. These results also carry over to pertubed Ornstein-Uhlenbeck processes.

The Markov chain approximation is a simulation scheme based on the symbol of the generator. In Chapter 6, we generalize this method to Feller processes with symbols whose drift and diffusion coefficient satisfy a linear growth condition. For affine processes we exploit the special structure of the state space to expand the linear growth condition to all coefficients of the symbol. Furthermore, our results carry over to more general state spaces as positive semidefinite matrices since we use an approach based on the symbol.

Index of Notation

This list is intended to aid cross-referencing, so notation that is specific to a single section is generally not listed. Some symbols are used locally, without ambiguity, in senses other than those given below.

General notation: Analysis

$a \vee b$, $a \wedge b$	$\max(a,b)$, $\min(a,b)$
a^+, a^-	$\max(a,0)$, $-\min(a,0)$
$\mathbf{i}$	imaginary unit
$\lvert x\rvert$	Euclidean vector and matrix norm
$\langle x,y\rangle = \sum_{k=1}^d x_k y_k$	for $x,y \in \mathbb{C}^d$
$e_\xi(x)$	$e^{\xi^\top x}$, $x, \xi \in \mathbb{C}^d$
$\operatorname{supp} f$	support, $\overline{\{f \neq 0\}}$
∇	gradient $\left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_d}\right)^\top$
∇^α	$\frac{\partial^{\alpha_1+\dots+\alpha_d}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$
$\mathcal{F}u = \hat{u}$	Fouier transform of a function u
$(T_t)_{t\geq 0}$	semigroup of operators
$(A, \mathcal{D}(A))$	generator
$q(x,D)$	pseudo-differential operator
$q(x,\xi)$	continuous negative definite symbol
$p(x,\xi)$	probabilistic symbol
S_d^+	space of $d \times d$ dimensional symmetric positive semidefinite matrices
S_d	space of $d \times d$ dimensional symmetric matrices
$\langle x,y\rangle = \mathrm{Tr}(xy)$	for $x,y \in S_d$

General notation: Probability

$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
$\sim$	'is distributed as'
a.s.	almost surely
$(X_t, \mathcal{F}_t)_{t\geq 0}$	adapted process
$B = (B_t)_{t\geq 0}$	Brownian motion
$L = (L_t)_{t\geq 0}$	Lévy process
ψ_L	characteristic exponent of a Lévy process $(L_t)_{t\geq 0}$
(l, Q, ν)	Lévy triplet
χ	truncation function
τ_r^x	$\inf\{t > 0;\ X_t \in \overline{B}^c(x,r)\}$

Sets and σ-algebras

A^c	complement of the set A
A°	open interior of the set A
$\overline{A}$	closure of the set A
$B(x,r)$	open ball, centre x, radius r
$\overline{B}(x,r)$	closed ball, centre x, radius r
$\mathcal{B}(D)$	Borel sets of D
$\mathcal{F}_t^X$	$\sigma(X_s \,:\, s \leq t)$

Spaces of measures and functions

$B(D)$	Borel functions on D
$B_b(D)$	— —, bounded
$C(D)$	continuous functions on D
$C_b(D)$	— —, bounded
$C_\infty(D)$	— —, $\lim\limits_{\lvert x\rvert\to\infty} u(x) = 0$
$C_c(D)$	— —, compact support
$C^k(D)$	k times continuously diff'ble functions on D
$C_\infty^k(D)$	— —, 0 at infinity (with their derivatives)
$C_c^k(D)$	— —, compact support

$\|u\|_\infty = \sup_x |u(x)|$ supremum norm
$\|u\|_{(k)}$ $\sum_{|\alpha|\le k} \|u\|_\infty$
$C^p_{\rho,\infty}$ weighted function space
$\|u\|_{(p),\rho} = \sum_{|\alpha|\le p} \|\rho D^\alpha u\|_\infty$ weighted norm
$L^p(D,\mu), L^p(\mu), L^p(D)$ L^p space w.r.t. the measure space $(D,\mathcal{F},\mu)$
$\|f\|_{L^p(\mu)}$ $\left(\int |f|^p \,\mathrm{d}\mu\right)^{1/p}$
$\mathcal{S}(\mathbb{R}^d)$ Schwartz space of rapidly decreasing smooth functions

Affine processes

$\mathbb{R}^m_+$ $\{x \in \mathbb{R}^m;\ x_i \ge 0\ \forall i = 1,\dots,m\}$
$\mathbb{C}^m_-$ $\{x \in \mathbb{C}^m;\ Re(x_i) \le 0\ \forall i = 1,\dots,m\}$
$D = \mathbb{R}^m_+ \times \mathbb{R}^n$ "canonical" state space
$d = m + n$
$I = \{1,\dots,m\}$ index set
$II = \{m+1,\dots,d\}$ index set
$x_I = (x_1,\dots,x_m)^\top$ projection on I coordinates for $x \in \mathbb{R}^m_+ \times \mathbb{R}^n$
$x_{II} = (x_{m+1},\dots,x_d)^\top$ projection on II coordinates for $x \in \mathbb{R}^m_+ \times \mathbb{R}^n$
F, R functional characteristics

Further abbreviations

CIR Cox-Ingersoll-Ross process
GOU generalized Ornstein-Uhlenbeck process
SDE stochastic differential equation