

Operator Triplets In Applied Mathematics

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Operator Triplets In Applied Mathematics

Integral Transforms and Fractional Integrals

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