Berichte aus der Mathematik

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Approximation and Regularity of Stochastic PDEs

Shaker Verlag
Aachen 2011

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.d-nb.de.

Zugl.: Dresden, Techn. Univ., Diss., 2011

Printed in Germany.

ISBN 978-3-8440-0172-3
ISSN 0945-0882

Shaker Verlag GmbH • P.O. BOX 101818 • D-52018 Aachen
Phone: 0049/2407/9596-0 • Telefax: 0049/2407/9596-9
Internet: www.shaker.de • e-mail: info@shaker.de

Approximation and Regularity of Stochastic PDEs (Abstract)

Felix Lindner

Stochastic partial differential equations (SPDEs, for short) constitute a wide and quickly growing area of research within mathematics, combining the fields of stochastic analysis and partial differential equations (PDEs, for short). In this thesis we study specific questions concerning the discretization and approximation of linear parabolic SPDEs and—intimately connected with it—the regularity of the solutions. SPDEs of parabolic type or stochastic evolution equations are usually treated from an abstract point of view as ordinary stochastic differential equations (SDEs, for short) in an infinite-dimensional state space. We are mainly interested in SPDEs with additive noise of the following form

$$dX_t = AX_t\,dt + G\,dM_t, \qquad t \in [0,T], \qquad x_0 \in H, \tag{1}$$

where H is a Hilbert space, A is the generator of a strongly continuous contraction semigroup on H, G is a linear operator and $M = (M_t)_{t\in[0,T]}$ is a martingale with values in H or in another Hilbert space U. We use the semigroup approach to SPDEs according to Da Prato, Zabczyk [2] and Peszat, Zabzcyk [7]. A weak solution $X = (X_t)_{t\in[0,T]}$ to equation (1) satisfies

$$\langle X_t, \zeta\rangle_H = \langle x_0, \zeta\rangle_H + \int_0^t \langle X_s, A^*\zeta\rangle_H\,ds + \langle GM_t, \zeta\rangle_H$$

for all $\zeta \in D(A^*)$, $t \in [0,T]$.

The thesis consists of two main parts, one of which—Chapter 2—is concerned with the so-called weak order of convergence of a space-time discretization scheme for an equation of type (1), where M is an infinite dimensional Lévy process. The investigations in the second part—Chapters 3 and 4—are motivated by the question whether adaptive and other nonlinear approximation methods for SPDEs pay off in the sense that they admit better convergence rates than uniform appoximation methods. In Chapter 3 we prove a result on the spatial regularity of the solution of an equation of type (1) within a certain scale of Besov spaces that is closely connected to the order of convergence of nonlinear approximation methods. In Chapter 4 we derive an explicit upper bound for the spatial Sobolev regularity of the solution, which restricts the order of convergence of uniform approximation methods. In what follows we give short summaries of the single chapters.

Chapter 1 provides the theoretical foundations, based on which the investigations in the following chapters are developed. In Section 1.1 we collect basic notations, definitions and results concerning function spaces, such as Sobolev-Slobodeckij and Besov Spaces. Essential notions concerning random variables and stochastic processes in Hilbert spaces are described in Section 1.2, and an introduction to stochastic integration in Hilbert spaces is given Section 1.3. In Section 1.4 we state basic results concerning SPDEs of parabolic type.

In Chapter 2 we study weak order of convergence of a (uniform) space-time discretization scheme for an equation of type (1). Usually, the driving process $M = (M_t)_{t\in[0,T]}$ is a Wiener process, and the majority of papers and books concerning SPDEs is restricted to such Gaussian noise models. There is, however, an increasing interest in more general, not necessarily Gaussian noise models, compare, e.g. Peszat, Zabczyk [7]. We consider an infinite dimensional non-Gaussian Lévy process as driving noise in (1). For a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ we investigate the following equation in $H = L_2(\mathcal{O})$,

$$dX_t = AX_t\,dt + Q^{1/2}\,dZ_t, \quad X_0 = x_0 \in H, \quad t \in [0,T]. \tag{2}$$

Here $Z = (Z_t)_{t\in[0,T]}$ is an impulsive cylindrical process and the operator Q describes the spatial covariance structure of the noise; we assume that $A^{-\alpha}$ has finite trace for some $\alpha > 0$ and that $A^\beta Q$ is bounded for some $\beta \in (\alpha-1,\alpha]$. A discretized solution $(X_h^n)_{n\in\{0,1,\ldots,N\}}$ is defined by the finite element method in space (parameter $h > 0$) and an implicit Euler scheme in time (parameter $\Delta t = T/N$). For $\varphi \in C_b^2(H;\mathbb{R})$ we derive an integral representation for the weak error $|\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_T)|$ and prove that

$$|\mathbb{E}\varphi(X_h^N) - \mathbb{E}\varphi(X_T)| = O(h^{2\gamma} + (\Delta t)^\gamma)$$

where $\gamma < 1-\alpha+\beta$. A similar discretization scheme has been studied by Debussche, Printems [3], but there the equation was driven by a cylindrical Wiener process. The main technical difference between [3] and our considerations lies in the fact that the impulsive cylindrical process Z is a purely discontinuous Hilbert space-valued martingale, while the cylindrical Wiener process is continuous. As a consequence, the main tools for deriving a suitable representation formula of the approximation error—the Itô formula and (connected with it) the backward Kolmogorov equations for certain processes associated with the solutions of the SPDE and their discretizations—are completely different in the paper [3] and for equation (2). The main task therefore is to find manageable expression for the approximation error, which allows estimates using techniques similar to those in [3]. We are able to obtain the same order of convergence as in the case of a cylindrical Wiener process. The results in this chapter are based on a joint paper with R.L. Schilling [6].

In Chapter 3 the scale of Besov spaces

$$B^\alpha_{\tau,\tau}(\mathcal{O}), \qquad \alpha > 0, \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}, \quad p \geq 2 \text{ fixed}, \tag{3}$$

is used to study the spatial regularity of the solutions of linear parabolic SPDEs with Gaussian noise on bounded Lipschitz domains $\mathcal{O} \subset \mathbb{R}^d$. The results in this chapter come from a cooperation within the research project 'Adaptive wavelet methods for SPDEs' of the DFG Priority Program 1324, see the joint paper with Cioica et al. [1]. It is well known that the smoothness of a target function $u \in L_p(\mathcal{O})$ within the Sobolev scale $W_p^s(\mathcal{O})$, $s \geq 0$, characterizes the convergence rate of uniform approximation schemes if the approximation error is measured in $L_p(\mathcal{O})$. The smoothness of $u \in L_p(\mathcal{O})$ in the Besov scale (3) determines the order of convergence that can be achieved by nonlinear approximation schemes, such as best n-term wavelet approximation. Roughly speaking, the equation we consider in Chapter 3 is of type (1), where $M = (M_t)_{t\in[0,T]}$ is a cylindrical Wiener process. However, the 'L_p theory for SPDEs on Lipschitz domains' by K.-H. Kim [5], which serves as a theoretical basis for our considerations, differs from the semigroup approach according to Da Prato, Zabczyk [2] and Peszat, Zabzcyk [7] considerably. Our result has the following structure: If

$$X \in L_p(\Omega \times [0,T], \mathcal{P}_T, \mathbb{P} \otimes \lambda; W_p^s(\mathcal{O}))$$

and if the the operator G is sufficiently regular, then

$$X \in L_\tau(\Omega \times [0,T], \mathcal{P}_T, \mathbb{P} \otimes \lambda;\ B^\alpha_{\tau,\tau}(\mathcal{O}))$$

for certain $\alpha > s$ and $1/\tau = \alpha/d + 1/p$. Here $(\Omega, \mathcal{A}, \mathbb{P})$ is the underlying probability space, $\mathcal{P}_T$ is the predictable σ-algebra and λ denotes Lebesgue measure on $[0,T]$. The proof is based on a combination of weighted Sobolev estimates and characterizations of Besov spaces by wavelet expansions. This result holds also for more general linear equations including, in particular, the case of multiplicative noise.

If the spatial regularity of $X = (X_t)_{t\in[0,T]}$ in the Sobolev scale $W^s_2(\mathcal{O}),\ s \geq 0$, is strictly smaller than the spatial regularity in the Besov scale $B^\alpha_{\tau,\tau}(\mathcal{O}),\ \alpha > 0,\ 1/\tau = \alpha/d + 1/2$—for instance due to singularities at the boundary—this indicates that nonlinear approximation w.r.t. the space variable really pays off. In Chapter 4 we show that under certain assumption this is indeed the case. We consider an equation of type (1) where $M = (M_t)_{t\in[0,T]}$ is a Wiener process, $x_0 = 0$ and $A : D(A) \subset L_2(\mathcal{O}) \to L_2(\mathcal{O})$ is the Laplace operator on a polygonal domain $\mathcal{O} \subset \mathbb{R}^2$ with zero Dirichlet boundary condition. Assuming sufficient spatial regularity of the driving noise, we prove that the solution process can be decomposed into a regular part with spatial L_2-Sobolev regularity of at least order 2 and a singular part whose spatial L_2-Sobolev regularity is restricted due to the shape of the domain $\mathcal{O}$. In the deterministic case this is a classical result by P. Grisvard [4]. The main task in the stochastic case is to handle the time-irregularity of $M = (M_t)_{t\in[0,t]}$. This irregularity has the consequence that the regular and the irregular part of the solution are $\mathbb{P}$-almost surely generalized functions w.r.t. the time variable t. Based on this decomposition we are able to prove that, for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$X(\omega) \notin L_2\big([0,T], \mathcal{B}([0,T]), \lambda; W^r_2(\mathcal{O})\big), \qquad \text{if } r > 1 + \frac{\pi}{\gamma_{\max}},$$

where $\gamma_{\max}$ is the largest interior angle at a corner of $\partial\mathcal{O}$. Once combined with the result of Chapter 3, we obtain examples where the spatial regularity in the Sobolev scale $W^s_2(\mathcal{O}),\ s \geq 0$, is strictly smaller that the regularity in the Besov scale $B^\alpha_{\tau,\tau}(\mathcal{O}),\ \alpha > 0,\ 1/\tau = \alpha/d + 1/2$.

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