Structural and Sensitivity Analysis for the Primal and Dual Problems in the Physical and Material Spaces

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Berichte aus der Mechanik

Daniel Materna

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Abstract

The present work is concerned with a complete and consistent representation of structural and sensitivity analysis for the primal and dual problems in structural mechanics. A complete description means that besides classical changes in the physical space (displacement space) also changes in the material space (geometry or design space) are allowed. This point of view yield a complexity of eight problems in structural mechanics. Based on a variational approach, these eight problems are completely derived in a continuous and finite element formulation for the model problem of nonlinear elasticity.

The above mentioned formulations are applied to mesh optimization (r-adaptivity) and shape optimization problems. First of all, in the framework of a complete primal problem, classical global r-adaptive mesh optimization strategies are considered and different error measures are introduced. Thereafter, a novel goal-oriented r-adaptive mesh optimization algorithm is proposed, in which the finite element mesh is optimized in such a way, that a chosen quantity of interest can be computed with high accuracy. Furthermore, shape optimization problems are investigated and the coherence to configurational mechanics is demonstrated.

Moreover, error estimators and improvement algorithms for first-order sensitivity relations are derived. Novel theorems for exact sensitivity relations for the state and a chosen quantity of interest are presented. These results are the basis for error estimators and for any improvement algorithms of design sensitivity relations. All of the required higher-order variations of the weak form of equilibrium are explicitly derived for shape design sensitivities. The efficiency and reliability of the error estimators and improvement algorithms are verified by means of numerical examples.

Kurzfassung

Die vorliegende Arbeit befasst sich mit der vollständigen und konsistenten Darstellung der Struktur- und Sensitivitätsanalyse für die primalen und dualen Probleme in der Strukturmechanik. Unter Vollständigkeit wird hier verstanden, dass neben den klassischen Veränderungen im physikalischen Raum (Verschiebungsraum) auch Veränderungen im materiellen Raum (Geometrie- oder Designraum) zulässig sind. Diese Betrachtungsweise führt zu einer Komplexität von acht Problemen in der Strukturmechanik. Basierend auf einem variationellen Zugang werden diese acht Probleme in der Arbeit vollständig kontinuierlich und diskret für das Modellproblem der nichtlinearen Elastizitätstheorie hergeleitet.

Die obigen Formulierungen werden anschließend auf Netzoptimierungsprobleme (r-Adaptivität) und Formoptimierungsprobleme angewendet. Zunächst werden im Rahmen eines vollständigen primalen Problems klassische globale r-adaptive Netzoptimierungsstrategien betrachtet und verschiedene Fehlermaße für r-Adaptivität eingeführt. Anschießend wird ein neuer zielorientierter r-adaptiver Netzoptimierungsalgorithmus entwickelt, mit dem ein gegebenes FE-Netz derart verbessert wird, dass eine gewählte Zielgröße möglichst genau berechnet werden kann. Ferner werden Formoptimierungsprobleme betrachtetet und die Beziehung zur Konfigurationsmechanik aufgezeigt.

Des Weiteren werden Fehlerschätzer und Verbesserungsalgorithmen für Sensitivitätsbeziehungen 1. Ordnung hergeleitet. Hierbei werden neue Theoreme für eine exakte Darstellung von Design-Sensitivitätsbeziehungen des Verschiebungsfeldes sowie für eine gewählte lokale Zielgröße formuliert. Diese bilden den Ausgangspunkt für Fehlerschätzer und Verbesserungsalgorithmen von Design-Sensitivitäten. Die hierfür benötigten höheren Variationen der schwachen Form des Gleichgewichts werden explizit für Formänderungen des materiellen Körpers bestimmt. Die Effizienz und Zuverlässigkeit der Fehlerschätzer und Verbesserungsalgorithmen wird anhand numerischer Beispiele verifiziert.

Preface

The research on the topic of the present thesis was performed during my time as a research assistant at the *Chair of Numerical Methods and Information Processing* at the *University of Dortmund* during the years 2004 to 2009.

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