Characters

in

Number Theory

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Preface

The homomorphisms of residue classes or their subgroups onto cyclic subgroups of real or complex numbers, which are called residue class characters, Dirichlet characters, or simply characters, make direct calculations possible; they are also a tool to select special terms in sums. This idea is especially used in the number theory of L-series, f.e. in the zeta function, the Gauss and Jacobi sums, and the proof of Dirichlet's theorem. Characters can be regarded as a step from basic to advanced number theory. Besides these residue class characters, more algebraic characters are defined.

With homomorphisms of finite Abelian groups, characters, the divisor function, von Mangoldt function, and Liouville function in Dirichlet series, Dirichlet series (general), Dirichlet L-series, Gauss , Ramanujan, Kloosterman, and Jacobi sums, and the proof of Dirichlet's theorem by different authors, a choice of applications of characters in number theory is given in this book.

I would appreciate discussions, remarks, and hints if there are mistakes.

Leimen, in October 2007 Uwe Kraeft

Choice of symbols

$\Rightarrow, \Leftarrow, \Leftrightarrow$	by this follows (in the given directions)
$\in$	is element of (is contained in)
$\notin$	is no element of (isn't contained in)
$A=\{a,b,c\}$	an example of a set A with elements a, b, and c
f, χ	mappings
$f(x)$	map of the original x
f^{-1}	inverse mapping
$a, \alpha, \dots$	elements
$-a, \alpha^{-1}$	inverse elements (with addition, multiplication)
$\# \{\dots\}$	number of elements
$m, n, \dots$	natural numbers
N	set of natural numbers 1, 2, 3, ... or any natural number
p	if not otherwise indicated, generally a prime
$\pi(n)$	the number of primes equal or less n
p_I, p_{II}	prime I: $4m+1$ with $m \in N$, prime II: $4n-1$ with $n \in N$
P	primes of N (P^1 with the unity)
N^0	$N \cup \{0\}$
N^-	$\{-n; n \in N\}$, set of negative integers -1, -2, -3, ...
Z	$=N \cup \{N^-\} \cup \{0\}$, ring of integers
Z/mZ	ring of congruence classes modulo m
Q	field of rational numbers a/b with $a \in Z$, $b \in N$
Q^+	set of positive rational numbers a/b with $a,b \in N$
R	field of real number algorithms
[...],]...[, ...	closed, open interval, ...
[a,b]	all x in the closed interval $a \le x \le b$
$a=\alpha+i\beta$	$=\alpha+\beta i$ complex number with $\alpha,\beta \in Q(R)$ and $i^2=-1$
$a^*=\bar{a}=\alpha-i\beta$	conjugate complex number
$a=\alpha+i\beta$	$=r(\cos\varphi + i*\sin\varphi)$ in polar coordinates;
	$r=\|a\|$ modulus or absolute value of a or length of vector $\underline{r}$;
	φ is called argument arg(a), arg a, or the phase angle
$C=C^1$	$=Q(i)$ or $R(i)$ field of complex numbers
Q(R,C)	Q or R or C
C^G	Gaussian integers with $(\alpha,\beta\neq 0) \in Z$
ζ	root of unity

$=$	equal (identical) by axioms or definitions
$\equiv$	$a \equiv b \pmod c \Leftrightarrow a \equiv b_c \Leftrightarrow (a-b)/c \in Z$ for $a,b \in Z$, $c \in N$
$\cong$	so near as you want but not identical
$\approx$	about, rounded, can f.e. be approximated for great n
$\sim$	similar order of magnitude
$\neq$	not equal
$<, >$	less, greater
$n!, (p-1)!$	product of all natural numbers equal or less n, (p-1)
$\binom{a}{b}$	$\frac{a!}{b!(a-b)!} = \frac{a(a-1)(a-2)*...*(a-b+1)}{b(b-1)(b-2)*...*1}$ binomial coefficient
$\sum_{i=1}^{n} a_i$	$=a_1+a_2+ \ldots +a_n$
$L(s,\chi)$	$=\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ Dirichlet L-series (see chapter 5; χ see chapter 1)
$\prod_{i=1}^{n} a_i$	$=a_1*a_2* \ldots *a_n$
a^n	nth power of a
a^{-n}	$1/a^n$
$a^{1/m}$	mth root of a (f.e. $a^{1/2}=\sqrt{a}$)
$a^{n/m}$	mth root of a^n
$\{a_i\}=(a_i)$	$\{a_1, a_2, \ldots, a_n\}$ sequence of numbers a_i
$b\|a$	b divides a, b is a factor (divisor) of a, $a \equiv 0_b$, $a \in Z-\{0\}$, $b \in N$
$p,(q>1) \in N$	p,q are natural numbers and $q>1$
$(a,b)=d$	gcd (see below) of a and b is d
$\{(a,b)\}=e$	lcm (see below) of a and b is e
a,b,c	triple of an equation's solution, f.e. PT (see below)
$\frac{p}{q}$	ratio or common fraction with $p \in Z$, $q \in N$, which can be proper $\left\|\frac{p}{q}\right\|<1$ or improper $\left\|\frac{p}{q}\right\| \geq 1$

$\left(\frac{a}{p}\right)=1$	$=(a'/p)=1 \Leftrightarrow a^{(p-1)/2}\equiv 1_p$ (Legendre's symbol)
$\left(\frac{a}{n}\right)$	$=(a'/p_1)(a'/p_2)(a'/p_3)*\ldots$ with $n=p_1p_2p_3*\ldots$, $(p_i>2)\in P$, and $(a,p_i)=1$ (Jacobi's symbol)
ln x	natural logarithm of x with $x=e^{\ln x}$
ind a	index of a with $a \equiv g^{\text{ind}\,a}$ (mod m), and (a,m)=1
*F**	multiplicative group (without zero) of a field *F*
f.e.	for example (e.g.)
gcd	greatest common divisor
lcm	least common multiple
modulus	divisor, from Latin "modulus" measure
PT	Pythagorean Triple
sgn z	$=\frac{z}{\lvert z\rvert}$ for $0 \neq z \in C(Q,R)$ or $=0$ if $z=0$, sign function
$\chi(n)$	character of $n \in N^0$ (see chapter 1)
$\varphi(n)$	Euler's function, see [Kr19 p. 186]
$\mu(n)$	Möbius function, see [Kr19 p. 187]
$\sigma_k(n)$	divisor function (see chapter 3.1)
$\Lambda(n)$	von Mangoldt function (see chapter 3.2)
$\lambda(n)$	Liouville function (see chapter 3.3)

Many other special symbols and number theoretic functions are explained in the text.

The order of this sequence of texts on number theory is twofold. The order following the date of printing is given at the end of this book. Another grouping is got by the colours of the covers after disciplines as follows:

arithmetic number theory:	light blue
sequences and series:	dark green
Diophantine Equations:	orange
algebraic number theory:	dark red
topological number theory:	purple
analytic number theory:	dark blue
statistical number theory:	light green
special numbers:	dark yellow
textbooks:	light yellow

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