A Multiscale Approximation for Operator Equations in Separable Hilbert Spaces — Case Study: Reconstruction and Description of the Earth's Interior

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Vom Fachbereich Mathematik der Universität Kaiserslautern
genehmigte Habilitationsschrift

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eingereicht: 5. Dezember 2001, Vollzug der Habilitation: 14. Mai 2002

Industriemathematik und Angewandte Mathematik

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D 386 (Habil.-Schr. Universität Kaiserslautern)

Shaker Verlag
Aachen 2002

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Michel, Volker:
A Multiscale Approximation for Operator Equations in Separable Hilbert Spaces - Case Study: Reconstruction and Description of the Earth´s Interior /
Volker Michel.
Aachen : Shaker, 2002
(Industriemathematik und Angewandte Mathematik)
Zugl.: Kaiserslautern, Univ., Habil.-Schr., 2002
ISBN 3-8322-0431-8

Printed in Germany.

ISBN 3-8322-0431-8
ISSN 1615-6390

Shaker Verlag GmbH • P.O. BOX 101818 • D-52018 Aachen
Phone: 0049/2407/9596-0 • Telefax: 0049/2407/9596-9
Internet: www.shaker.de • eMail: info@shaker.de

Preface

The increasing interest in advanced methods in signal and image processing motivated the development of euclidean wavelets (see, for example, [42]). For more details on euclidean wavelets and their application in signal and image processing see, for instance, Mallat's presentation at the International Congress of Mathematicians 1998 ([43]) and the references herein. Later on the advantages of wavelets became of interest for geomathematical problems such as the representation of geopotentials. Here scalar functions on the two–dimensional sphere have to be approximated at different scales.
The shape of the Earth and the particular data situations made the development of further new kinds of wavelets necessary. After the development of scalar spherical wavelets in [39] a generalized concept of scalar wavelets on regular surfaces was introduced ([34]), allowing a multiscale determination of solutions of (exterior) boundary value problems. A series of further publications (see e.g. [20], [21]) treated the wavelet analysis of scalar functions on closed surfaces. Relevant applications of those methods are, for instance, the modelling of the gravitational potential, the absolute value of the magnetic field on the Earth's surface or the topography. The obtained scale and detail spaces are usually not orthogonal. Only recently, non–bandlimited orthogonal wavelets on the sphere have been discovered ([29]).
Often the acquired data are used to recover quantities, on which those data depend. An example of such an inverse problem is the calculation of the gravitational potential at the Earth's surface from measurements at the satellite orbit. This inversion is discontinuous, such that a regularization is necessary, which has been realized by wavelets ([36], [52]).
The interest in the application of the new multiscale concept to further geoscientifical problems required an extension of the theory and the methods to functions with different domains or different values. The inverse problem of the reconstruction of the Earth's mass density distribution from gravitational data needs, for example, the treatment of functions which are defined on the whole Earth and not only the surface. Moreover, on this three–dimensional domain, we may not assume that the functions of interest can be approximated by harmonic polynomials, in contrast to the situation of surface problems. In the 3D case the L^2–space can be decomposed into the set of harmonic functions and an orthogonal space of so–called anharmonic functions, where only the harmonic part of the density distribution can be recovered from gravitational data. This particular situation

requires two multiscale concepts: one for the regularization of the discontinuous recovery of the harmonic part of the solution ([30], [44], [45], [47]) and one for the determination of the anharmonic part from additional (non–gravitational) data ([30]).
Present satellite missions, such as CHAMP, GRACE, and GOCE, offer huge amounts of gravitational and magnetic data of the Earth. Since the magnetic field is available in terms of vector data, the development of spherical vector wavelets ([6], [8], [9], [11]) became necessary. Moreover, the satellite GOCE will offer (components of) the Hesse tensor of the gravitational potential. Therefore, spherical tensor wavelets were also developed ([21], [49]).
Furthermore, pyramid schemes are available as fast numerical algorithms (see [54] for the bandlimited case, [20] for the non–bandlimited case and [11] for the generalization to vector fields including error control) and multiscale signal–to–noise thresholding methods are applicable to error–affected data (see [17], [18] for the euclidean case and [24], [25], [32] for the spherical case). For more details on spherical wavelets and their application to geomathematical problems we refer to [20], [21], and the references in the monographs.

From the uncertainty principle (see, for example, [26]) for the approximation of functions on a sphere or the inner space of a sphere, here called ball, we know that a restriction to one particular point in space and one particular frequency within the same model is not possible, where the ‘frequency’ stands here for the degree of a homogeneous harmonic polynomial on the unit sphere and is also called ‘momentum’ in physical literature. A classical Fourier analysis of a signal, which is here a function on a sphere or a ball, consists of the calculation of L^2–scalar–products of the signal and the homogeneous harmonic polynomials. This is one of the extremal cases of the uncertainty principle. In the Fourier analysis we fix single frequencies with the effect that the space localization is completely lost, since we have to integrate over the whole domain. The contrary extremal situation yields the inverse Fourier transform where all Fourier coefficients, multiplied with the polynomials, are summed up in the Fourier series. A pointwise evaluation (which is anyway not possible in L^2–spaces and requires, for example, the Lipschitz continuity of the function (see [40])) of the signal is done at the expense of the frequency localization, since the contributions of all frequencies have to be summed up.
A trade–off between both extremal situations is represented by product kernels. Here, we will concentrate on kernels with variable localizations, where the different levels of localizations are represented by a parameter called scale. This approach allows us to zoom into different levels of resolution. Such multiscale techniques use product kernels of the (more or less) general form

$$K(x,y) = \sum_{n=n_0}^{\infty} \sum_{j=1}^{j_n} K^\wedge(n) a_{n,j}(x) \otimes b_{n,j}(y) \tag{1}$$

for (almost every) $x \in C$ and $y \in D$ in given domains C and D. Here $\{a_{n,j}\}_{(n,j)}$ and $\{b_{n,j}\}_{(n,j)}$ represent complete orthonormal systems in given Hilbert spaces of functions. Since the functions can also be vectorial or tensorial, we used the tensor product here, which certainly is the usual scalar multiplication if at least one of the functions is a scalar function. If we now have a function G with the Fourier series

$$G = \sum_{n=n_0}^{\infty} \sum_{j=1}^{j_n} G^\wedge(n,j) b_{n,j}, \tag{2}$$

then we obtain

$$K * G := \int_D K(.,y) G(y)\, dy = \sum_{n=n_0}^{\infty} \sum_{j=1}^{j_n} K^\wedge(n) G^\wedge(n,j) a_{n,j}. \tag{3}$$

If $K^\wedge(n)$ vanishes for all n with the exception of one value $n = M$ (with $j_M = 1$), then we have again the case of a Fourier transform that concentrates on one frequency $n = M$ and determines

$$G^\wedge(M,1) = \int_D G(y) b_{M,1}(y)\, dy \tag{4}$$

by integrating over the whole domain. On the other hand, if $K^\wedge(n) = 1$ for all $n \geq n_0$ then we have the case of the so–called Dirac kernel, which has to be understood in the distributional sense. Here $K * G = G$, such that we obtain the maximal space localization and completely lose the frequency localization (inverse Fourier transform).

If we now want a trade–off between space and frequency localization we have to choose coefficients $K^\wedge(n)$ somewhere between those two extremal situations. The, at first sight, most simple way is to define $K^\wedge(n) = 1$ for $n < N$ and $K^\wedge(n) = 0$ for $n \geq N$. The obtained kernel is called Shannon scaling function. It is a simple version of a low pass filter. The lower N is, the smaller is the band–width of the low pass filter and the more frequency localizing is the kernel. By increasing N we decrease the frequency localization and increase the space localization. Therefore, it appears to be reasonable to construct a whole class of product kernels $\{\Phi_J\}_J$ by defining, for example, $N = 2^J$, in order to be able to look at the considered functions at different resolutions, just like varying the enlargement factor of a spyglass.

Of course, there are various other possible ways of constructing such families of low pass filters, which are called scaling functions, in this context. Therefore, we require only a few conditions on the kernels, such that the Shannon scaling function represents only a special case of a large class of possible choices. We distinguish two kinds of scaling functions: bandlimited and non–bandlimited. In case of a bandlimited kernel K only a finite number of coefficients $K^\wedge(n)$ is non–vanishing. The obtained convolutions $K*G$ are finite linear combinations of the functions $a_{n,j}$, which are, for example, polynomials. In view of the uncertainty principle, bandlimited kernels are more frequency localizing than non–bandlimited ones and, consequently, less space localizing. It, therefore, depends on the considered problem whether a bandlimited or a non–bandlimited approach is more appropriate. Note that the calculations, that have to be done, are theoretically also integrations over the whole domain,

$$K * G = \int_D K(.,y)G(y)\,dy, \tag{5}$$

as in the computation of the Fourier coefficients. However, the values $y \mapsto K(x,y)$ of a space localizing kernel K are only in the neighbourhood of x essentially different from zero in contrast to the values of the (polynomial) functions $b_{n,j}$ that are used for the determination of the Fourier coefficients $\langle G, b_{n,j}\rangle_Y$. This property of the product kernels has the important advantage that we only have to use a small subset of the available data $G(y_n)$ for the numerical calculation of the integrals by setting $K(x,y) = 0$ if the distance of x and y is sufficiently large. The increased space localizing behaviour of non–bandlimited kernels is, therefore, not only attractive if phenomena in small areas are investigated, but is also helpful if huge data sets, like the millions of data available from present and future satellite missions, have to be treated, since then the domain of integration, where $K(x,.)$ is not set to 0, can be further minimized.

If we have a closer look at the multiscale concepts that were developed for the various types of scalar, vectorial and tensorial functions on different domains for a series of direct and inverse problems, we discover that primarily the proofs use typical properties of a Hilbert space, such as the Parseval identity and the relation between strong and weak convergence. The intention of this work is, therefore, to show that it is really possible to construct a multiresolution theory and to prove all characteristic properties of a multiscale method by merely using the Hilbert space axioms and its implications. It is even not necessary to assume that the elements of the Hilbert spaces are functions. We will, therefore, discuss the general equation $TF = G$, where $T : X \to Y$ is a linear, continuous, injective operator from the Hilbert space X into the Hilbert space Y, where we know a spectral representation (singular value decomposition) $Ta_{n,j} = T^\wedge(n) b_{n,j}$ with respect to complete orthonormal systems $\{a_{n,j}\}_{n\in\{n_0,n_0+1,\ldots\},\,j\in\{1,\ldots,j_n\}}$ and $\{b_{n,j}\}_{n\in\{n_0,n_0+1,\ldots\},\,j\in\{1,\ldots,j_n\}}$ in X and Y, respectively, where the knowledge of the spectral representation is certainly a restriction to the applicability of the method, but is, for example, given for all approaches listed above. In our formulation of the problem we include, in particular, direct problems of approximating given functions by allowing $T = \mathrm{Id}$, $X = Y$. Moreover, ill–posed inverse problems, such as the reconstruction of the Earth's density distribution from gravitational data, and well–posed inverse problems, such as boundary value problems, are also examples of such an operator equation.
Our method allows the calculation of an approximation F_J to the exact solution $F = T^{-1}G$ at each scale J, where in the limit of the scale the approximation converges to the exact solution. Moreover, the method automatically yields a regularization, i.e. the approxima-

tions continuously depend on the right hand side G, even if T^{-1} is not continuous, and even for perturbations $\mathcal{E} \in Y$ with $G + \mathcal{E} \notin \operatorname{im} T$ we can calculate approximations to $T^{-1}G$ out of $G + \mathcal{E}$. This means that the concept, introduced in this work, allows to treat a large class of direct and inverse problems, such that it is no more necessary to develop multiscale techniques for each problem of the above mentioned type separately. We will construct a scale continuous and a scale discrete multiresolution and show that they can be transformed into one another. Moreover, the pyramid schemes for numerical purposes will be generalized to our Hilbert space context, and we will also formulate an abstract multiscale denoising concept.

After the description of the general Hilbert space concept of multiresolution analysis, we discuss a particular problem, namely the reconstruction of the Earth's density distribution. Although Jules Verne already described a 'Journey to the Centre of the Earth' in his novel, nobody has ever managed to reach depth of more than a few kilometres up to now. Since this situation will probably not essentially change at least for a long time, we have to use surface–based, airborne or spaceborne information sources that depend somehow on properties of the Earth's interior. Therefore, any investigation of the Earth's interior is always connected with an inverse problem.
The knowledge of the density distribution of the Earth is of great importance for many reasons. The modelling of global and local structures enables a better understanding of phenomena like earthquakes and vulcano eruptions. Moreover, a high resolution of the density distribution in certain regions without the need of drilling offers an efficient way of searching for rare resources like oil. Finally, even in archaeology modern technologies are used to discover hidden buildings by the analysis of measurements made at the surface. Those examples of applications show that the density distribution is interesting at different levels of resolution, i.e. in different spatial extensions. Global models are needed as well as continental models and more localized models even up to dimensions of a few meters as it is the case, for example, in archaeology. This is only one reason why multiscale methods appear to be the most appropriate tools for the reconstruction and the description of the Earth's density distribution.
The available data for the reconstruction of the Earth's interior can be separated into two types: gravitational and non–gravitational. For an exemplary historical overview of the determination of the density distribution from gravitational data see [45]. The calculation of the density function F out of gravitational measurements is based on the inversion of Newton's gravitational potential

$$\int_{\mathrm{Earth}} \frac{F(x)}{|x - .|}\, dx. \tag{6}$$

The available data types (will) include potential values on the Earth's surface and first and second derivatives of the potential at a satellite orbit. Those quantities are derivable from the German satellite CHAMP (<u>CHA</u>llenging <u>M</u>ini–Satellite <u>P</u>ayload), launched in 2000, the GFZ/NASA 'two satellite configuration' GRACE (Gravity Recovery And Climate Experiment), which will launch in 2002, and the ESA satellite GOCE (Gravity Field and Steady–State OCEan Circulation Mission, see [19]), which will launch in 2006. For mathematical aspects of the determination of the first and the second radial derivative we refer, for example, to [28] and [31]. The described data situation implies that actually more than one inverse problem must be solved. Fortunately, a singular value decomposition of the kind described above is known for each relevant inverse problem in this context.

According to Hadamard an inverse problem is called well–posed if and only if the solution exists, is unique and continuously depends on the right hand side. Otherwise, the problem is called ill–posed. The reconstruction of the density distribution from gravitational data is ill–posed, no matter if the data are given in the form of the potential or its first or second radial derivative and no matter if those data are available at the surface of the Earth or in the outer space (airborne or spaceborne data). This ill–posedness is caused by a violation of each of Hadamard's three criteria.
The existence of the solution is not always given. Although we will show in this work that the images of the considered operators T are dense in Y, measuring errors might cause a right hand side which is not an element of the image $\operatorname{im} T$ and, therefore, yields an unsolvable problem. However, the application of our general method allows us to calculate an approximation to the solution $F = T^{-1}G$ of the unperturbed problem $TF = G$ out of a perturbed right hand side $\tilde{G}$, even if $\tilde{G} \notin \operatorname{im} T$. Moreover, our multiscale denoising concept enables us to improve the approximations by eliminating data that are dominated by noise.
The stability of the reconstruction, i.e. the continuous dependence of the solution on the right hand side, is not given in any of the above described inverse problems. This instability is well–known for the case of given potential data at the surface (see, for example, [45]) and can easily be extended to the other investigated data types, as we will show in this work. This violation of Hadamard's third criterion requires a regularization method. A classical truncated singular value decomposition is certainly one possible way of regularizing the problem. However, a truncated singular value decomposition is only appropriate if globals trends are analysed. For the investigation of local phenomena high frequency contributions also have to be taken into account. Since the instability is reflected here in the multiplication of high frequency Fourier coefficients with large numbers $(T^\wedge(n))^{-1}$ that diverge as the frequency n tends to infinity, we have a (too) strong sensitivity of the inversion to errors in high frequency Fourier coefficients. If those coefficients shall not be set to 0, which would be the effect of a truncated singular value decomposition, then they must be multiplied with a (positive) factor $\Phi_J^\wedge(n)$ that converges faster to 0 than $(T^\wedge(n))^{-1}$ diverges as $n \to \infty$, such that the obtained approximations F_J to $F = T^{-1}G$ continuously depend on G. Our method will provide both possibilities, the bandlimited and the non–bandlimited. Moreover, it has the property that F_J tends to F in the limit

$J \to \infty$ (or $J \to 0+$, depending on the chosen scale).
From [58] and the further treatments in, for example, [2], [4], [30], and [45] we know that the gravitational potential only depends on the harmonic part of an L^2–density–function. The orthogonal anharmonic part has the potential 0 and can, therefore not be reconstructed. We will show in this work that the null space is the same in case of the radial derivatives as right hand sides. This non–uniqueness property is an essential violation of Hadamard's second criterion, since the space of harmonic polynomials of degree $\leq n$ has the dimension $(n+1)^2$, whereas the space of anharmonic polynomials of degree $\leq n$, $n \geq 2$, has the dimension $n^3/6 - n/6$ (see [45]). Moreover, from [45] we know that a radially symmetric density distribution, which is, from the global point of view, approximately true in case of the mantle and the core of the Earth, has a constant harmonic part, such that such structures can only be modelled by including anharmonic functions into the calculations. Therefore, we also have to use non–gravitational data for a complete model of the Earth's density distribution.
The non–gravitational data for the reconstruction of the Earth's density are primarily available from seismic observations but can also come from other sources such as geomagnetism and geoelectricity. Seismic data are, for example, the travel times of earthquakes, that depend on the velocity distribution of the compressional and the shear waves in the Earth's interior and the surface waves at the Earth's surface (see, for instance, [1], [15], and the references therein), where the velocities are related to the density (see, for example, [12] and [61]). Another seismic data source for the investigation of the Earth's interior is given by the eigenoscillations caused by large earthquakes (see, for instance, [10], [14], and [46]). Further data sources are imaginable. It is not the intention of this work to investigate all possible types of inversions of non–gravitational data. Therefore, we assume that such inversions have already been done, such that we know the density at a finite point grid, which in particular has the advantage that we are able to mix density data derived from various non–gravitational data sources. Our method allows the calculation of the anharmonic part of the Earth's density distribution at different scales.

With our approach it is now, among various other applications, possible to construct a multiscale description of the Earth's mass density distribution. Moreover, the combination of the approximate, regularized, reconstruction of the harmonic part, derived from surface–based, airborne or spaceborne gravitational measurements, with the anharmonic part, calculated from a priori data, generates an improved density model in the sense that we are able to include data from gravitational and non–gravitational sources into the calculations.

First of all, I thank Prof. Dr. Willi Freeden for encouraging me to continue my research at the University of Kaiserslautern, for spending his rare available time on my questions, and for giving me the best possible support. Moreover, I thank all present and former members of the Geomathematics Group for valuable discussions and new ideas. Mentioning some of them would diminish the contribution of the others.

Volker Michel

Contents

Preface **i**

1 Basic Fundamentals **1**
1.1 Metric, Banach and Hilbert spaces . . . 1
1.2 Legendre Polynomials . . . 9
1.3 Jacobi Polynomials . . . 11
1.4 Spherical Harmonics . . . 12
1.5 Decomposition of L^2–functions on balls . . . 15

2 Multiresolution Analysis for Operator Equations **19**
2.1 Formulation of the General Problem . . . 20
2.2 Product Kernels and Convolutions . . . 25
2.3 Low Pass and Band Pass Filters . . . 31
2.3.1 Scale Continuous Case . . . 31
2.3.2 Scale Discrete Case . . . 32
2.4 Multiresolution Analysis . . . 42
2.4.1 Scale Continuous Case . . . 42
2.4.2 Scale Discrete Case . . . 53
2.5 An Error Bound . . . 60
2.6 A Pyramid Scheme . . . 61
2.7 Noise Reduction . . . 64

3 Determination of the Earth's Density Distribution **73**
3.1 Reconstruction from Gravitational Data . . . 74
3.1.1 From the Potential at the Surface . . . 74
3.1.2 From the Potential at a Satellite Orbit . . . 80
3.1.3 From the First Radial Derivative at an Orbit (CHAMP and GRACE mission) . . . 83
3.1.4 From the Second Radial Derivative at an Orbit (GOCE mission) . . . 85
3.1.5 Regularization Error . . . 87
3.2 Determination of the Anharmonic Part . . . 91

4 Conclusions and Outlook **95**

A Appendix: Illustration of the Harmonic Density **97**

Bibliography **103**

Index **109**