

Stabilized Mixed Triangular and Tetrahedral Finite Elements with Volume and Area Bubble Functions

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Abstract

In order to improve the poor performance of linear triangular and tetrahedral elements, such as locking and stress oscillations, this work presents a successful stabilization method for mixed finite element formulations. The method of incompatible modes and the enhanced strain method are considered for the stabilization of the mixed finite element formulation. In addition to volume bubble functions, new area bubble functions are presented. In contrast to the classical static condensation scheme of SIMO and RIFAI, constant stabilization matrices are introduced. This stabilization scheme is considered for physically non-linear problems in the case of small deformations and for hyperelasticity in the case of large deformations. In the numerical examples, the mixed formulations for the method of incompatible modes and the enhanced strain method, involving volume and area bubble functions, are compared to different element formulations familiar from literature. For both stabilized triangular and tetrahedral elements, the results are excellent in the convergence study for the displacements, and locking of incompressible materials is completely avoided. Furthermore, stabilized mixed triangular and tetrahedral elements achieve drastic damping behavior of stress oscillatory effects. Both mixed finite elements with area bubbles, the method of incompatible modes and the enhanced strain method, show better results than mixed finite elements with volume bubbles. Moreover, in all numerical examples no significant differences between the method of incompatible modes and the method of enhanced strains could be observed.

Zusammenfassung

In dieser Arbeit wird eine erfolgreiche Stabilisierung der gemischten Finite-Elemente-Formulierung vorgestellt, wobei diese in der Lage ist die schlechten Eigenschaften linearer Dreiecks- und Tetraeder-Elemente, wie "locking" und Spannungszusillationen, zu verbessern. Für die Stabilisierung wird die Methode der Inkompatiblen Moden und die erweiterte Dehnungsmethode berücksichtigt. Zusätzlich zu den Volumen-Bubble Funktionen werden neue Flächen-Bubble Funktionen vorgestellt. Im Gegensatz zur klassischen statischen Kondensation nach SIMO und RIFAI werden konstante Stabilisierungsmatrizen eingeführt. Diese Stabilisierungsmethode wird für physikalisch und geometrisch nicht-lineare Probleme angewendet. In den numerischen Beispielen werden die gemischten stabilisierten Elemente mit Finite-Elemente-Formulierungen aus der Literatur verglichen. Sowohl für die Dreiecke als auch für die Tetraeder ist eine deutliche Verbesserung zu erkennen. Zwischen der Methode der inkompatiblen Moden und der erweiterten Dehnungsmethode sind die Unterschiede von marginaler Bedeutung. Jedoch werden mit den Flächen-Bubble Funktionen bessere Ergebnisse erzielt als mit den Volumen-Bubble Funktionen. Volumen "locking" lässt sich vollständig vermeiden. Des Weiteren können die starken Spannungszusillation drastisch gedämpft werden.

Preface

This work was carried out between 2005 and 2010 at the Chair of Engineering Mechanics (LTM) at the University of Paderborn, Germany. The research work was funded by the Deutsche Forschungsgemeinschaft (DFG) under the grants MA 1979/5-2 and MA 1979/8-1 within the joint research project: “Thermomechanical modeling and characterization of the solid-liquid interactions in casting processes”.

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Paderborn, Dezember 2010

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