

Exponentials, Diffusions, Finance, Entropy and Information

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To

Monika, Julian and Leana

Hilde, Rupert and Helmut

*Centa, Geli, Fritz, Linus, Lea,
Anna, Alexander and Victoria*

Kai and Elias

ℰ in loving memory of Christine and Walter

Preface

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