

# **Robust Numerical Algorithms Based on Corrected Operator Splitting for Two-Phase Flow in Porous Media**

Von der Fakultät Mathematik und Physik der Universität Stuttgart  
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Naturwissenschaften (Dr. rer. nat.) genehmigte Abhandlung

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Berichte aus der Mathematik

**Yufei Cao**

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# Memorial for Prof. Magne S. Espedal

It was with great shock and sadness that I received the news that Prof. Magne S. Espedal passed away on January 25, 2010. Prof. Espedal was born in Norway in 1942, studied at the University of Bergen, Norway in 1962, and was employed at the Department of Mathematics, University of Bergen in 1971. In 1990, he was promoted to a full professor.

Prof. Espedal was crucial in establishing a research group and educational programs in reservoir mechanics in the Department of Mathematics. He was one of the main forces behind the creation of the interdisciplinary Center for Integrated Petroleum Research (CIPR) in Bergen, which is a national center of excellence. He also had many leading roles in Norwegian research, including the Norwegian Research Council. To many researchers, he was regarded as an informal leader of the Norwegian community in applied mathematics.

Prof. Espedal's scientific work covered a wide range of topics, and his research focus was the mathematical modeling of flow and transport processes in porous media. He had written several influential papers, often in close collaboration with graduate students and international colleagues on different subjects including operator splitting methods, domain decomposition methods and upscaling. Lately, he had been working on modeling microbial processes for enhanced oil recovery.

As one of my co-supervisors, I owe Prof. Espedal deep gratitude for his patient supervision. He kindly hosted me at CIPR, University of Bergen for three months and for another short visit afterwards. I am very grateful for his hospitality and constructive discussions on my work, especially the work done in Chapter 4 which is related to one of the research fields he had been working on. I would like to express my heartfelt condolence to his family. I dedicate this work to Prof. Espedal.

As a friendly mentor, I will miss him dearly.

Yufei Cao

*Stuttgart, April 2010*



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Yufei Cao

*Stuttgart, April 2010*



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# Notation

The following table shows the significant symbols used in this work. Local notations are explained in the text.

Symbol	Definition	Dimension
<b>Greek Letters:</b>		
$\alpha$	Van Genuchten parameter	[ Pa <sup>-1</sup> ]
$\Gamma_D$	Dirichlet boundary part of $\partial\Omega$	[ - ]
$\Gamma_N$	Neumann boundary part of $\partial\Omega$	[ - ]
$\gamma$	angle deformation	[ rad ]
	parameter needed in Van Genuchten relative permeability function	[ - ]
	angle between the streamline direction and the normal vector on a cell edge	[ ° ]
$\gamma_{ij}$	cell facet between cells $K_i$ and $K_j$	[ - ]
$\Delta t$	time step size in the case without using operator splitting	[ s ]
	splitting step in the case using operator splitting	[ s ]
$\Delta t_a$	inner time step of the advection equation in operator splitting	[ s ]
$\Delta t_d$	inner time step of the diffusion equation in operator splitting	[ s ]
$\Delta x, \Delta y$	discretization lengths in $x$ - and $y$ -direction	[ m ]
$\varepsilon$	parameter needed in Van Genuchten relative permeability function	[ - ]
	dimensionless scaling factor	[ - ]
$\theta$	contact angle	[ ° ]
$\lambda$	Brooks-Corey parameter (pore size distribution index)	[ - ]
$\lambda_\alpha$	mobility of phase $\alpha$	[ (ms)/kg ]
$\lambda_t$	total mobility	[ (ms)/kg ]

$\mu$	dynamic viscosity	[ kg/(ms) ]
$\mu_\alpha$	dynamic viscosity of phase $\alpha$	[ kg/(ms) ]
$\nu$	scaled normal vector	[ m ]
$\rho$	density	[ kg/m <sup>3</sup> ]
$\rho_\alpha$	density of phase $\alpha$	[ kg/m <sup>3</sup> ]
$\sigma$	interfacial tension	[ N/m <sup>2</sup> ]
$\tau$	shear stress	[ N/m <sup>2</sup> ]
	time-of-flight along a streamline	[ s ]
$\phi$	porosity	[ - ]
$\phi_i$	standard nodal basis function for node $x_i$ of $\mathcal{T}_h$	[ - ]
$\chi$	characteristic function	[ - ]
$\Psi$	slope limiter function	[ - ]
$\psi_i$	new basis function for interior node $x_i$ of $\mathcal{T}_h$	[ - ]
$\Omega$	solution domain	[ - ]
$\partial\Omega$	boundary of domain $\Omega$	[ - ]

**Latin Letters:**

$C, C_1, C_2$	generic constants	[ - ]
$C(\Omega)$	linear space of continuous functions	[ - ]
$\mathcal{C}$	one-dimensional coarse grid	[ - ]
$F_a$	numerical advective flux	[ m <sup>2</sup> /s ]
$F_d$	numerical diffusive flux	[ m <sup>2</sup> /s ]
$\mathcal{F}$	one-dimensional fine grid	[ - ]
<b>G</b>	gravity term	[ kg/(m <sup>2</sup> s <sup>2</sup> ) ]
$H$	discretization length of the coarse grid $\mathcal{C}$	[ m ]
$H^{-s}(\Omega)$	dual space of $H^s(\Omega)$ , $s > 0$	[ - ]
$\mathcal{H}_h$	discrete solution operator of the diffusive step	[ - ]
<b>I</b>	identity matrix	[ - ]
$\mathcal{I}_h, \hat{\mathcal{I}}_h, \mathcal{I}_h^*, P_h$	interpolation operators	[ - ]
$\hat{\mathcal{I}}_{\Gamma_N}, \mathcal{I}_{\Gamma_N}^*$	traces of $\hat{\mathcal{I}}_h, \mathcal{I}_h^*$ on Neumann boundary $\Gamma_N$	[ - ]
$\mathcal{J}$	partition of time interval	[ - ]
$K$	grid cell of MPFA mesh (or control volume for cell-centered finite volume method)	[ - ]
$\partial K$	boundary of grid cell or control volume $K$	[ - ]
<b>K</b>	intrinsic permeability	[ m <sup>2</sup> ]
<b>K<sub>α</sub></b>	effective permeability of phase $\alpha$	[ m <sup>2</sup> ]
$L$	number of inner time steps $\Delta t_d$	[ - ]
$L^p(\Omega), H^s(\Omega)$	Sobolev spaces ( $p = 2, \infty, s > 0$ )	[ - ]
$\mathcal{L}$	space differential operator	[ - ]
$\mathcal{L}_a$	advection operator	[ - ]

$\mathcal{L}_d$	diffusion operator	[ - ]
$N$	number of time/splitting steps	[ - ]
$\mathcal{N}_h$	node index set of $\mathcal{T}_h$	[ - ]
$P$	global pressure	[ Pa ]
$\mathcal{P}_c^f, \hat{\mathcal{P}}_c^f$	prolongation operators from $\mathcal{C}$ to $\mathcal{F}$	[ - ]
$\mathbf{R}$	rotation matrix	[ - ]
$\mathcal{R}_f^c$	restriction operator from $\mathcal{F}$ to $\mathcal{C}$	[ - ]
$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^d$	Euclidean spaces	[ - ]
ST	simplified L triangle	[ - ]
$S_\alpha$	saturation of phase $\alpha$	[ - ]
$S_{\alpha r}$	residual saturation of phase $\alpha$	[ - ]
$S_e$	effective saturation	[ - ]
$\mathcal{S}_h$	discrete solution operator of the advective step	[ - ]
T	L triangle	[ - ]
$T$	simulation time	[ s ]
	temperature	[ °C ]
$\mathcal{T}_h$	finite element mesh	[ - ]
$\hat{\mathcal{T}}_h$	MPFA mesh (partition of domain $\Omega$ )	[ - ]
$\mathcal{T}_h^*$	dual mesh of $\mathcal{T}_h$	[ - ]
$V$	control volume for vertex-centered finite volume method	[ - ]
$V_h$	finite element space	[ - ]
$V_h^*$	dual volume element space	[ - ]
$e$	edge	[ - ]
	relative discrete $L^2$ norm of the error	[ - ]
$f, \bar{f}, \hat{f}, \tilde{f}$	fluxes through a certain edge	[ m <sup>2</sup> /s ]
$f_\alpha$	fractional flow function of phase $\alpha$	[ - ]
$g$	(scalar) gravity	[ m/s <sup>2</sup> ]
$\mathbf{g}$	gravity vector $(0, 0, -g)^T$	[ m/s <sup>2</sup> ]
$g_N$	Neumann boundary value	[ m/s ]
$h$	maximum diameter of all grid cells	[ m ]
	discretization length of the fine grid $\mathcal{F}$	[ m ]
$h_K$	diameter of grid cell $K$	[ m ]
$k$	scalar intrinsic permeability	[ m <sup>2</sup> ]
$k_{r\alpha}$	relative permeability of phase $\alpha$	[ - ]
$l$	length of line segment	[ m ]
$m$	Van Genuchten parameter	[ - ]
$n$	Van Genuchten parameter	[ - ]
$\mathbf{n}$	unit normal vector	[ - ]
$p, \bar{p}$	pressure	[ Pa ]
$p_\alpha$	pressure of phase $\alpha$	[ Pa ]
$p_c$	capillary pressure	[ Pa ]

$p_d$	entry pressure	[ Pa ]
$q$	source/sink	[ 1/s ]
$q_\alpha$	source/sink of phase $\alpha$	[ 1/s ]
$r$	radius	[ m ]
	saturation gradient ratio	[ - ]
$s$	streamline (nodes)	[ - ]
$s$	arc length along a streamline	[ m ]
$t$	time	[ s ]
	transmissibility coefficient	[ (m <sup>3</sup> s)/kg ]
$\mathbf{t}$	unit tangential vector	[ - ]
$u$	unknown of a differential equation	[ - ]
$v^\perp$	normal velocity	[ m/s ]
$\mathbf{v}$	Darcy velocity	[ m/s ]
$\mathbf{v}_\alpha$	phase velocity of phase $\alpha$	[ m/s ]
$\mathbf{v}_{\alpha m}$	modified phase velocity of phase $\alpha$	[ m/s ]
$\mathbf{v}_a$	average velocity	[ m/s ]
$\mathbf{v}_t$	(Darcy) total velocity	[ m/s ]
$\mathbf{v}_{ta}$	average total velocity	[ m/s ]
$\mathbf{w}$	front velocity	[ m/s ]
$\mathbf{w}_a$	average front velocity	[ m/s ]
$x, \bar{x}, x', x'', y$	points in the Euclidean space	[ - ]
$x_c$	shock collision point	[ - ]

**Subscripts:**

$\alpha$	phase, either wetting ( $w$ ) or non-wetting ( $n$ )
$K$	grid cell
$c$	envelope
$h$	mesh size
$n$	non-wetting phase
$res$	residual flux
$w$	wetting phase

**Superscripts:**

$D$	Dirichlet boundary
$L$	left side
$N$	Neumann boundary
$R$	right side
$d$	number of dimensions
$l$	inner time step
$n$	time/splitting step

