

RESIDUATED STRUCTURES WITH INVOLUTION

Vom Fachbereich Mathematik
der Technischen Universität Darmstadt
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)
genehmigte

Dissertation

von

Dipl.-Math. Annika Meike Wille
aus Kassel

Referent:	Prof. Dr. C. Herrmann
Koreferent:	Prof. Dr. D. Mundici
Koreferent:	Prof. Dr. M. Otto
Koreferent:	Prof. Dr. C. Tsinakis
Tag der Einreichung:	8. Juni 2006
Tag der mündlichen Prüfung:	21. Juli 2006

Darmstadt 2006

D17

Berichte aus der Mathematik

Annika Meike Wille

Residuated Structures with Involution

D 17 (Diss. TU Darmstadt)

Shaker Verlag
Aachen 2006

Bibliographic information published by the Deutsche Nationalbibliothek

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at <http://dnb.d-nb.de>.

Zugl.: Darmstadt, Techn. Univ., Diss., 2006

Copyright Shaker Verlag 2006

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior permission of the publishers.

Printed in Germany.

ISBN-10: 3-8322-5789-6

ISBN-13: 978-3-8322-5789-7

ISSN 0945-0882

Shaker Verlag GmbH • P.O. BOX 101818 • D-52018 Aachen

Phone: 0049/2407/9596-0 • Telefax: 0049/2407/9596-9

Internet: www.shaker.de • e-mail: info@shaker.de

*Dedicated to my parents,
Meike and Friedrich Wille.*

ACKNOWLEDGMENTS

At this point I would like to thank those who, in some way or other, contributed to the development of this thesis.

I would like to thank my advisor Christian Herrmann very much for his dedication in the last years. He always had time for giving suggestions and advice, listening to talks for conferences, and explaining algebraic contexts to me. Sometimes we sat for hours in front of a blackboard (which was actually green in our case) thinking about beautiful mathematics.

I would also like to thank very much Constantine Tsinakis who has been like an advisor and mentor to me since the time when I visited Vanderbilt University in Nashville, Tennessee. He introduced me to the subject of residuated lattices and made it possible that I could visit Vanderbilt University several times, where we worked on mathematical problems together.

During conferences I met Peter Jipsen, who I could also visit at Chapman University, California, for two weeks in fall 2003. He spent a lot of time with me talking about residuated lattices and I learnt plenty from him. I would like to thank him very much.

I also would like to thank Christian Herrmann, Daniele Mundici, Martin Otto, and Constantine Tsinakis for refereeing my thesis.

I am grateful to many fellow PhD students or members of my research group AG₁ for listening to mathematical thoughts, encouraging each other, or just drinking coffee together. I would like to name Nikolaos Galatos, who graduated at Vanderbilt University in 2003, Niklas Niemann, my “Doktorbruder”, Achim Blu-

mensath, who is (besides being a very good mathematician) an expert in \LaTeX and font design, Ian Wood with whom I started to study at Technische Universität Darmstadt, and Jochen Hechler.

During the past three years, I have gotten financial support from universities and foundations who I would like to thank very much. In 2003 I got a stipend from Vanderbilt University, from January 2004 to January 2005 Technische Universität Darmstadt supported me with a stipend. Furthermore, I got a fellowship from Studienstiftung des Deutschen Volkes (German National Academic Foundation). From February 2005 Deutsche Telekom Stiftung accepted me as a scholarship holder. I would like to thank Deutsche Telekom Stiftung not only for financial support, but also for making it possible for me to participate in interesting conferences and seminars.

I am most grateful for the support and guidance of my family, in particular of my mother, Meike Wille. She was the first person who saw my fascination in mathematics and encouraged me, together with my father, Friedrich Wille. Especially in the last years I am thankful for her council and assistance.

Finally, I had the fortune to meet a wonderful person, Thomas. I would like to thank him very much for his understanding and constant support.

CONTENTS

1. INTRODUCTION	1
2. PRELIMINARIES	5
2.1. Universal algebra	5
2.1.1. First-order structures, terms and formulas	5
2.1.2. Lattices	7
2.1.3. Congruences and homomorphisms . . .	8
2.1.4. Reduced products and ultraproducts . .	11
2.1.5. Varieties and quasivarieties	12
2.1.6. Free algebras	13
2.1.7. Elementary classes and finite axiomatiz- ability	13
2.1.8. Subdirect products	15
2.1.9. Jónsson's Lemma	16
2.1.10. Equational theory	16
2.2. Residuated lattices	22
2.2.1. Definition	22
2.2.2. Basic properties	24
2.2.3. Structure theory of \mathcal{RL}	26
3. POINTED RESIDUATED LATTICES	35
3.1. Weakly involutive residuated lattices	36
3.2. Involutive residuated lattices	37
3.3. Basic properties of weakly involutive residuated lattices	39

3.4.	Examples of weakly involutive residuated lattices	41
3.4.1.	Finite examples	41
3.4.2.	Boolean algebras	46
3.4.3.	Lattice-ordered groups	47
3.4.4.	Bounded generalized MV-algebras	48
3.4.5.	MV-algebras	56
3.4.6.	Reducts of relation algebras	57
3.5.	Subvariety lattice of \mathcal{WInRL}	60
3.6.	Structure theory of $p\mathcal{RL}$ and its subvarieties	61
3.7.	Fully invariant normal filters of pointed residuated lattices	64
3.8.	Dedekind-MacNeille completion	66
4.	MINIMAL VARIETIES OF $In\mathcal{RL}$	69
4.1.	The atomic structure of $\mathbf{L}(\mathcal{RL})$	69
4.2.	From modules to involutive residuated lattices	71
4.3.	An uncountable family of involutive residuated lattices	77
4.4.	The atomic structure of $\mathbf{L}(In\mathcal{RL})$	81
5.	A LATTICE EMBEDDING INTO INVOLUTIVE RESIDUATED LATTICES	89
5.1.	A lattice embedding into involutive residuated lattices	90
5.2.	About the structure of $\mathbf{L}(In\mathcal{RL})$	95
5.3.	Finite height conjecture	97
5.4.	Modular involutive residuated lattices	97
5.4.1.	Decidability problems of certain subvarieties of $In\mathcal{RL}$	98
5.4.2.	Undecidability of the equational theory of $ModIn\mathcal{RL}$	99

5.5.	Finite axiomatizability of various involutive residuated lattice varieties	99
5.5.1.	Preservation by ultraproducts	99
5.5.2.	Application of modular lattice results	102
5.5.3.	Further consequences	104
6.	A GENTZEN SYSTEM FOR INVOLUTIVE RESIDUATED LATTICES	107
6.1.	Involutive normal form	108
6.2.	Girard quantales	109
6.3.	Gentzen system	110
6.4.	Soundness and completeness	113
7.	THE FEP FOR INVOLUTIVE RESIDUATED LATTICES	125
7.1.	Preliminaries	125
7.2.	The finite embeddability property for IRL	126
7.3.	The finite embeddability property for $IInRL$	128
7.4.	Failure of the FEP for $InRL$	130
8.	THE WORD PROBLEM FOR VARIOUS RESIDUATED LATTICE VARIETIES	133
8.1.	Preliminaries	133
8.2.	Presentations and the word problem	134
8.3.	The interpretation technique	136
8.4.	Generalization of group results	138
8.5.	Expanding the power set monoid of a group	141
8.6.	Conclusion	143
A.	CONSTRUCTING FINITE EXAMPLES WITH GAP	145
A.1.	Program files	145
A.2.	Selected functions	146